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OMEGA SUBGROUPS OF POWERFUL p-GROUPS

BY

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ABSTRACT

Let G be a powerful finite p-group. In this note, we give a short elementary proof of the following facts for all $i \geq 0$: (i) $\exp \Omega_i(G) \leq p^i$ for odd p, and $\exp \Omega_i(G) \leq 2^{i+1}$ for $p = 2$; (ii) the index $|G : G^{p^i}|$ coincides with the number of elements of G of order at most p^i.

Let G be a finite p-group. For every $i \geq 0$, denote by $\Omega_{\{i\}}(G)$ the set of elements of G of order at most p^i, and let $\Omega_i(G)$ be the subgroup it generates. On some occasions, $\Omega_i(G)$ coincides with $\Omega_{\{i\}}(G)$, in other words, the exponent of $\Omega_i(G)$ is at most p^i. This happens, of course, for abelian groups, and also for powerful p-groups if p is odd, as shown by L. Wilson in [6]. In his Ph.D. Thesis [5], Wilson also obtained that $\exp \Omega_i(G) \leq 2^{i+1}$ for $p = 2$, which is best possible. On the other hand, Héthelyi and Lévai [3] have proved that $|G : G^{p^i}| = |\Omega_{\{i\}}(G)|$ for all powerful p-groups. These properties of powerful p-groups have been generalized by González-Sánchez and Jaikin-Zapirain [2] to normal subgroups lying inside G^2.

The aim of this note is to provide a short direct proof of the results of Wilson and Héthelyi–Lévai. The proof is completely elementary and only uses basic properties of powerful p-groups (as can be found in Chapter 2 of [1] or Chapter 11 of [4]), and the following version of Hall's collection formula: if x and y are elements of a group G, p is a prime and $n \in \mathbb{N}$, then

$$(xy)^{p^n} \equiv x^{p^n} y^{p^n} \pmod{\gamma_2(H)^{p^n} \gamma_p(H)^{p^{n-1}} \cdots \gamma_{p^n}(H)}$$

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and

$$[x,y]^{p^n} \equiv [x^{p^n}, y] \pmod{\gamma_2(K)^{p^n}\gamma_p(K)^{p^{n-1}}\cdots\gamma_{p^n}(K)},$$

where $H = \langle x, y\rangle$ and $K = \langle x, [x,y]\rangle$.

It is convenient to make the following convention in order to write our results more easily: if G is a p-group and $x \in G$, let us define the meaning of the inequality $o(x) \le p^i$ with $i < 0$ (which is actually impossible to hold) to be that $x = 1$. Define accordingly $\Omega_i(G) = \Omega_{\{i\}}(G) = 1$ for all $i < 0$, which is coherent with the definition given for $i \ge 0$.

Theorem 1: *Let G be a powerful p-group. Then, for every $i \ge 0$:*

(i) *If $x, y \in G$ and $o(y) \le p^i$, then $o([x,y]) \le p^i$.*

(ii) *If $x, y \in G$ are such that $o(x) \le p^{i+1}$ and $o(y) \le p^i$, then $o([x^{p^j}, y^{p^k}]) \le p^{i-j-k}$ for all $j, k \ge 0$.*

(iii) *If p is odd, then $\exp\Omega_i(G) \le p^i$.*

(iv) *If $p = 2$, then $\exp\Omega_i(T) \le 2^i$ for any subgroup T of G which is cyclic over G^2. In particular, $\exp\Omega_i(G^2) \le 2^i$.*

Proof: We apply induction on the order of G globally to all the assertions of the theorem. Thus we prove that (i) through (iv) hold for a group G under the assumption that they hold for all powerful groups of smaller order. Of course, we may suppose that G is non-cyclic.

(i) The commutator $[x,y] = (y^{-1})^x y$ is a product of two elements of order $\le p^i$ in the subgroup $T = \langle y, G'\rangle$. If p is odd, then T is a proper powerful subgroup of G, by Lemma 11.7 of [4]. If $p = 2$, then $H = \langle y, G^2\rangle$ is also powerful and proper in G, and T is a subgroup of H that is cyclic over H^2. Since (iii) and (iv) hold for powerful groups of order smaller than that of G, the result follows.

(ii) In this case, we also apply reverse induction on i. If we write $\exp G = p^e$, then the result is clear for $i \ge e$: just note that $[x^{p^j}, y^{p^k}] = g^{p^{j+k}}$ for some $g \in G$, since G is powerful. Suppose now that $i < e$ and put $T = \langle x, [x, y^{p^k}]\rangle$. By Hall's collection formula,

$$[x^{p^j}, y^{p^k}] \equiv [x, y^{p^k}]^{p^j} \pmod{\gamma_2(T)^{p^j}\gamma_p(T)^{p^{j-1}}\cdots\gamma_{p^j}(T)}. \tag{1}$$

Since $o([x, y^{p^k}]) \le o(y^{p^k}) \le p^{i-k}$, it follows that $[x, y^{p^k}]^{p^j} \in \Omega_{i-j-k}(T)$. Note that, arguing as in the last paragraph, the induction on the group order yields that $\exp\Omega_n(T) \le p^n$ for all n. (In particular $\exp T \le p^{i+1}$.) Thus if we prove that all the subgroups appearing in the modulus of congruence (1) lie in $\Omega_{i-j-k}(T)$, then we can conclude that the order of $[x^{p^j}, y^{p^k}]$ is at most p^{i-j-k}, as desired.

First, since $\gamma_2(T) = \langle [x, y^{p^k}, x] \rangle^T \le \Omega_{i-k}(T)$, we get $\gamma_2(T)^{p^j} \le \Omega_{i-j-k}(T)$. Let us now prove, by induction on r, that $\gamma_{r+2}(T) \le \Omega_{i-k-r}(T)$ for all $r \ge 0$. We have just seen this for $r = 0$. In the general case, it suffices to show that if $a \in \gamma_{r+1}(T)$ and $b \in T$ then $o([a, b]) \le p^{i-k-r}$. Since $\gamma_{r+1}(T) \le \Omega_{i-k-r+1}(T)$, we know that $o(a) \le p^{i-k-r+1}$. On the other hand, $a \in \gamma_{r+1}(T) \le [G^{p^k}, G, \overset{r+1}{\ldots}, G]$ and G being powerful imply that $a = g^{p^{k+r+1}}$ for some $g \in G$, and then $o(g) \le p^{i+2}$. But $b \in T$ has order $\le p^{i+1}$, so the reverse induction on i we are applying yields that $o([a, b]) = o([g^{p^{k+r+1}}, b]) \le p^{i-k-r}$.

Now if either p is odd and $r \ge 1$ or if $p = 2$ and $r \ge 2$, we have $\gamma_{p^r}(T) \le \gamma_{r+2}(T)$, and consequently $\gamma_{p^r}(T)^{p^{j-r}} \le \Omega_{i-j-k}(T)$. Thus we only have to check that $\gamma_2(T)^{2^{j-1}} \le \Omega_{i-j-k}(T)$ for $p = 2$. This follows by observing that $[x, y^{2^k}, x] = [g^{2^{k+2}}, x]$ for some g of order $\le 2^{i+2}$ and hence, by the reverse induction on i, this commutator has order at most 2^{i-k-1}.

(iii) Let $x, y, z \in G$ be elements of order p. Then $[x, y] = g^p$ for some g of order at most p^2, and it follows from (ii) that $[x, y, z] = 1$. Hence the nilpotency class of $\Omega_1(G)$ is at most 2. Since p is odd, $\Omega_1(G)$ is then regular and $\exp \Omega_1(G) \le p$. For the general case, by the induction on the group order, we get $\exp \Omega_{i-1}(G/\Omega_1(G)) \le p^{i-1}$. This, together with $\exp \Omega_1(G) \le p$, yields that $\exp \Omega_i(G) \le p^i$.

(iv) As in (iii), it is enough to prove that $\exp \Omega_1(T) \le 2$. (We need $T \trianglelefteq G$ in order to apply the induction hypothesis to $G/\Omega_1(T)$, but this is not a problem, since we may assume $G^2 \le T$.) For this purpose, we are going to see that $\Omega_1(T)$ is abelian, that is, that any two elements $a, b \in T$ of order 2 commute. If either a or b lie in G^2, this is a consequence of (ii), so we suppose that $a, b \in \Omega_1(T) \setminus G^2$. Since T is cyclic over G^2, it follows that $b = av$ for some $v \in G^2$, and we have to prove that a and v commute. Now $v^2 = (ab)^2 = [b, a] = [v, a] \in G^8$, so there exists $u \in G^4$ such that $u^2 = v^2$. We also deduce that v^2 has order at most 2, whence $o(u) = o(v) \le 4$. It is then a consequence of (ii) that u and v commute. Put $w = v^{-1}u \in G^2$. Since $w^2 = 1$ it follows again from (ii) that w commutes with a and, of course, it also commutes with v. Hence $(au)^2 = (avw)^2 = (av)^2 = 1$ and, since $a, au \in \langle a, G^4 \rangle$, we may apply the induction on the group order to deduce that a and au commute. Since $au = avw$ we conclude that a commutes with v, as desired. ∎

Corollary 2: *Let G be a powerful 2-group. Then:*

(i) $\exp [\Omega_i(G), G] \le 2^i$.

(ii) $\exp \Omega_i(G) \le 2^{i+1}$.

Proof: (i) By part (i) of Theorem 1, $[\Omega_i(G), G]$ can be generated by elements of order at most 2^i. Since G is powerful, it follows that $[\Omega_i(G), G] \le \Omega_i(G^2)$, and then we only have to apply part (iv) of Theorem 1.

(ii) As observed above, we have $[\Omega_i(G), G] \le \Omega_i(G^2)$, and consequently the factor group $\Omega_i(G)/\Omega_i(G^2)$ is abelian. But this group can be generated by elements of order at most 2, hence its exponent is also at most 2. Since $\exp \Omega_i(G^2) \le 2^i$, we conclude that $\exp \Omega_i(G) \le 2^{i+1}$. ∎

Some other properties of omega subgroups of powerful p-groups can be immediately deduced from Theorem 1, and their proof is left to the reader. For example, that for odd p the subgroup $\Omega_1(G^p)$ is powerful and that $\exp \gamma_{r+2}(\Omega_i(G)) \le p^{i-r}$ for all $r \ge 0$. In particular, the nilpotency class of $\Omega_i(G)$ is at most $i+1$ for $p > 2$.

Finally, we prove the result of Héthelyi and Lévai. We need the following lemma, which is an immediate consequence of Hall's collection formula.

LEMMA 3: *Let G be a powerful p-group of exponent p^e. Then for every $0 \le i \le e-1$, and every $x \in G$, $y \in G^{p^{e-i-1}}$, we have $(xy)^{p^i} = x^{p^i} y^{p^i}$.*

THEOREM 4: *Let G be a powerful p-group. Then $|G : G^{p^i}| = |\Omega_{\{i\}}(G)|$ for all $i \ge 0$.*

Proof: We argue by induction on the group order. Let $\exp G = p^e$. We only need to consider the case when $i \le e-1$. On the other hand, by the previous lemma, the map $x \mapsto x^{p^{e-1}}$ is a homomorphism from G onto $G^{p^{e-1}}$, and by the first isomorphism theorem, $|\Omega_{\{e-1\}}(G)| = |G : G^{p^{e-1}}|$.

So we assume in the remainder that $i \le e-2$. Set $\overline{G} = G/G^{p^{e-1}}$. By the induction hypothesis, $|G : G^{p^i}| = |\overline{G} : \overline{G}^{p^i}| = |\Omega_{\{i\}}(\overline{G})|$. Hence if $X = \{x \in G \mid x^{p^i} \in G^{p^{e-1}}\}$ then $|G : G^{p^i}| = |X|/|G^{p^{e-1}}|$. Thus it suffices to prove that $|X| = |\Omega_{\{i\}}(G)||G^{p^{e-1}}|$.

Note that $x \in X$ if and only if there exists $y \in G^{p^{e-i-1}}$ such that $x^{p^i} = y^{p^i}$ and, by the previous lemma, this is equivalent to $xy^{-1} \in \Omega_{\{i\}}(G)$. Thus $X = \Omega_{\{i\}}(G)G^{p^{e-i-1}}$ is the union of all the cosets of elements of $\Omega_{\{i\}}(G)$ with respect to the subgroup $G^{p^{e-i-1}}$, and in order to get the cardinality of X we have to find out how many different cosets of that kind there are. Let $gG^{p^{e-i-1}}$ be a coset with $g \in \Omega_{\{i\}}(G)$. Again by the lemma, the elements of order at most p^i in that coset are exactly those in $g(\Omega_{\{i\}}(G) \cap G^{p^{e-i-1}}) = g\Omega_{\{i\}}(G^{p^{e-i-1}})$, so their number is $|\Omega_{\{i\}}(G^{p^{e-i-1}})|$, which is independent of the choice of g. In other words, the elements of $\Omega_{\{i\}}(G)$ are evenly distributed in the different cosets with respect to the subgroup $G^{p^{e-i-1}}$. Hence the number of cosets we wanted is $|\Omega_{\{i\}}(G)|/|\Omega_{\{i\}}(G^{p^{e-i-1}})|$ and

$$|X| = \frac{|\Omega_{\{i\}}(G)|}{|\Omega_{\{i\}}(G^{p^{e-i-1}})|}|G^{p^{e-i-1}}|. \tag{2}$$

Finally, since $i \leq e-2$, it follows that $G^{p^{e-i-1}}$ is a proper powerful subgroup of G. By the induction hypothesis, we have $|\Omega_{\{i\}}(G^{p^{e-i-1}})| = |G^{p^{e-i-1}} : G^{p^{e-1}}|$, and we get from (2) that $|X| = |\Omega_{\{i\}}(G)||G^{p^{e-1}}|$, as desired. ∎

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